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Torsion, Scalar Field, Mass and FRW Cosmology

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In the Einstein–Cartan space U_4, an axial vector torsion together with a scalar field connected to a local scale factor have been considered. By combining two particular terms from the $SO(4, 1)$ Pontryagin density and then modifying it in a $SO(3, 1)$ invariant way, we get a Lagrangian density with Lagrange multipliers. Then under FRW-cosmological background, where the scalar field is connected to the source of gravitation, the Euler–Lagrange equations ultimately give the constancy of the gravitational constant together with only three kinds of energy densities representing mass, radiation and cosmological constant. The gravitational constant has been found to be linked with the geometrical Nieh-Yan density.

KEY WORDS: Torsion; Nieh-Yan density; gravitational constant; FRW cosmology.
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1. INTRODUCTION

At present, standard cosmology starts with two basic assumptions: (i) at sufficiently large scale, matter distribution is spatially homogeneous and isotropic; and (ii) the large-scale structure of the universe can be described by Einstein's theory of gravity. The geometrical evolution of the universe can then be determined by Einstein's equations where the energy momentum tensor acts as the source. The Friedmann–Robertson–Walker (FRW; 1922, 1929, 1935) universe is so far the most provocative and important cosmological model of the universe. It is also one of the simplest. It is isotropic, spatially homogeneous, and fluid-filled. The FRW models serve as an introduction to the study of homogeneous models. A FRW universe admits a six-parameter group of isometries whose surfaces of transitivity are space-like three surfaces of constant curvature. Minkowski space, de Sitter space and anti-de Sitter space are all special cases of the general FRW spaces (Hawking and Ellis, 1973). When several noninteracting sources are present in the universe, the total energy momentum tensor that appears on the right-hand side of the

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Einstein's equation will be the sum of the energy momentum tensor for each of the sources. Spatial homogeneity and isotropy imply that the energy momentum tensor for the ith source is diagonal and has the form $\overset{(i)}{T}{}_{\beta}{}^{\alpha} = \text{dia}[\,\rho_i,\ -p_i,\ -p_i,\ -p_i\,]$. Here ρ_i and p_i are, respectively, the energy density and the pressure for the ith source which obey the energy conservation law $d(\rho_i a^3) = -p_i d(a^3)$, where $a(t)$ is the radius of the universe at time t. The evolution of the energy density of each component is essentially dependent on the parameter $\omega_i \equiv \frac{p_i}{\rho_i}$. In particular, $\omega_i = 0$, $\frac{1}{3}$ or -1, respectively, for non-relativistic mass density, radiation density or vacuum energy density (Padmanabhan, 2003).

It is well known that if we add the cosmological constant as the only source of curvature in Einstein's equation, the resulting spacetime is highly symmetric and has an interesting geometrical structure. In particular, in the case of positive cosmological constant, we get the well known de Sitter manifold (Padmanabhan, 2003).

Kibble (1961) and Sciama (1962) pointed out that the *Poincaré* group, which is the semi-direct product of translation and Lorentz rotation, is the underlying gauge group of gravity and found the so-called Einstein–Cartan theory where mass-energy of matter is related to the curvature and spin of matter is related to the torsion of spacetime. One major drawback of *Poincaré* group is that it is a non–semi-simple group, which implies that there is no Lagrangian yielding its Yang–Mills equations (Aldrovandi and Kraenkel, 1988). There exists a general procedure (Aldrovandi and Kraenkel, 1989) to check whether or not a set of field equations leads to a coherent theory, i.e. a theory that can be quantized. If we apply it to Yang–Mills equations for non–semi-simple groups, we find that they are never consistent. Here we see that though the *Poincaré* group is the classical group for relativistic kinematics, it cannot be given a quantum version. Now by minimal addition of extra terms this inconsistent theory can be transformed to a good theory and we find a Lagrangian of a gauge theory for a semi-simple group, the de Sitter group (Aldrovandi and Pereira, 1988). In this way, the de Sitter gauge theory comes up as the corrected *Poincaré* gauge theory. Alternatively, there are other approaches where de Sitter group based Yang–Mills theories are shown to be producing either Ashtekar formulation of gravity (Nieto *et al.*, 1994) or Einstein–Cartan version of general relativity (Botta Cantcheff, 2002).

It is a remarkable result of differential geometry that certain global features of a manifold are determined by some local invariant densities. These topological invariants have an important property in common—they are total divergences and in any local theory these invariants, when treated as Lagrangian densities, contribute nothing to the Euler–Lagrange equations. Hence, in a local theory only few parts, not the whole part, of these invariants can be kept in a Lagrangian density. Recently, in this direction, a gravitational Lagrangian has been proposed (Mahato, 2002), where a Lorentz invariant part of the de Sitter Pontryagin density has been treated as the Einstein–Hilbert Lagrangian. By this way, the role of torsion in the underlying manifold has become multiplicative rather than additive one and the Lagrangian looks like *torsion* $\otimes$ *curvature*. In other words, the additive torsion is decoupled from the theory but not the multiplicative one. This indicates that torsion is uniformly nonzero everywhere. In the geometrical sense, this implies that microlocal spacetime is such that at every point there is a direction vector (vortex line) attached to it. This effectively corresponds to the noncommutative geometry having the manifold $M_4 \times Z_2$, where the discrete space Z_2 is just not the two-point space (Connes, 1994) but appears as an attached direction vector. In this paper, we shall try to establish the 'constancy' of the gravitational constant under the background of a scalar field ϕ which is either localized at laboratory scale or connected to the local universal scale factor of an isotropic and homogeneous universe and, in particular, also try to derive the power law of the cosmic energy density with respect to the local scale factor.

2. PONTRYAGIN DENSITY, SCALAR FIELD AND GRAVITY LAGRANGIAN

Cartan's structural equations for a Riemann–Cartan spacetime U_4 are given by (Cartan, 1922, 1924)

$$T^a = de^a + \omega^a{}_b \wedge e^b \tag{1}$$

$$R^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b, \tag{2}$$

here $\omega^a{}_b$ and e^a represent the spin connection and the local frames respectively.

In U_4 there exists two invariant closed four forms. One is the well-known Pontryagin (Chern and Simons, 1974, 1971) density P and the other is the less-known Nieh-Yan (1982) density N given by

$$P = R^{ab} \wedge R_{ab} \tag{3}$$

and

$$\begin{aligned} N &= d(e_a \wedge T^a) \\ &= T^a \wedge T_a - R_{ab} \wedge e^a \wedge e^b. \end{aligned} \tag{4}$$

The minimal Lagrangian density of a spin$-\frac{1}{2}$ field ψ with an external gravitational field with torsion, is given by (Mielke, 2001)

$$L_D = \frac{i}{2}\{\overline{\psi^*}\gamma \wedge D\psi + \overline{D\psi} \wedge^* \gamma\psi\} +^* m\overline{\psi}\psi - \frac{1}{4}A \wedge \overline{\psi}\gamma_5{}^*\gamma\psi, \tag{5}$$

where the exterior covariant derivative D is torsion free, A is the axial vector part of the torsion two form, $\gamma = \gamma_\mu\, dx^\mu = \gamma_a e^a$ and * is the Hodge duality operator.

Therefore, considering the source in the matter Lagrangian, we can simply assume that the torsion is given by an axial vector only.

In presence of axial vector torsion, one naturally gets the Nieh-Yan density from (4)

$$N = -R_{ab} \wedge e^a \wedge e^b = -^*N\eta, \tag{6}$$

where

$$\eta := \frac{1}{4!}\epsilon_{abcd}e^a \wedge e^b \wedge e^c \wedge e^d \tag{7}$$

is the invariant volume element. It follows that $^*N$, the Hodge dual of N, is a scalar density of dimension (length)$^{-2}$.

We can combine the spin connection and the vierbeins multiplied by a scalar field together in a connection for $SO(5, 1)$, in the tangent space, in the form

$$W^{AB} = \begin{bmatrix} \omega^{ab} & \phi e^a \\ -\phi e^b & 0 \end{bmatrix}, \tag{8}$$

where $a, b = 1, 2, \ldots, 4; A, B = 1, 2, \ldots, 5$ and ϕ is a variable parameter of dimension $(length)^{-1}$ and corresponds a local length scale. In some earlier works (Chandia and Zanelli, 1977; Mahato, 2002, 2004), ϕ has been treated as an inverse length constant. With this connection, we can obtain $SO(4, 1)$ Pontryagin density as

$$\begin{aligned} F^{AB} \wedge F_{AB} &= R^{ab} \wedge R_{ab} + 2\phi^2 d(e^a \wedge T_a) + 4\phi d\phi \wedge e^a \wedge T_a \\ &= P + dC_{T\phi}, \end{aligned} \tag{9}$$

where

$$C_{T\phi} := 2\phi^2 e^a \wedge T_a, \tag{10}$$

$$P := -R^a{}_b \wedge R^b{}_a = -\left(\bar{R}^a{}_b \wedge \bar{R}^b{}_a + 2\bar{R}^a{}_b \wedge \hat{R}^b{}_a + \hat{R}^a{}_b \wedge \hat{R}^b{}_a\right), \tag{11}$$

$$\bar{R}^b{}_a = d\bar{\omega}^b{}_a + \bar{\omega}^b{}_c \wedge \bar{\omega}^c{}_a, \tag{12}$$

$$\hat{R}^b{}_a = dT^b{}_a + \bar{\omega}^b{}_c \wedge T^c{}_a + T^b{}_c \wedge \bar{\omega}^c{}_a + T^b{}_a \wedge T^c{}_a \tag{13}$$

and

$$T^a_b = \omega^a{}_b - \bar{\omega}^a{}_b \quad \text{s.t.} \quad T^a_b \wedge e^b = T^a \tag{14}$$

Now $-\bar{R}^a{}_b \wedge \bar{R}^b{}_a$, the purely Riemannian torsion-less part of P, is a closed four form and is given by

$$-\bar{R}^a{}_b \wedge \bar{R}^b{}_a = -d\left(\bar{\omega}^a{}_b \wedge \bar{R}^b{}_a - \frac{1}{3}\bar{\omega}^a{}_b \wedge \bar{\omega}^b{}_c \wedge \bar{\omega}^c{}_a\right) = dC_R \tag{15}$$

where

$$C_R = -\left(\bar{\omega}^a{}_b \wedge \bar{R}^b{}_a - \frac{1}{3}\bar{\omega}^a{}_b \wedge \bar{\omega}^b{}_c \wedge \bar{\omega}^c{}_a\right).$$

With the hypothesis that only the axial vector part of the torsion is present in the physical world, we can write

$$\begin{aligned} T^a &= e^{e\mu} T_{\mu\nu\alpha} dx^\nu \wedge dx^\alpha, \\ T^{ab} &= e^{e\mu} e^{b\nu} T_{\mu\nu\alpha} dx^\alpha \end{aligned} \tag{16}$$

and

$$^*A = T = \frac{1}{3!} T_{\mu\nu\alpha} dx^\mu \wedge dx^\nu \wedge dx^\alpha \quad \text{s.t.} \quad N = 6dT \tag{17}$$

In this framework, we see that

$$\hat{R}^a{}_b \wedge \hat{R}^b{}_a = -2d\left(A \wedge dA - \frac{1}{3} T^a{}_b \wedge T^b{}_c \wedge T^c{}_a\right) = -dC_T \tag{18}$$

and

$$2\bar{R}^a{}_b \wedge \hat{R}^b{}_a = -4\mathcal{R} dT + 8\mathcal{R} d^{ab} \bar{\nabla}(A_b \eta_a) = 8d(G^{ab} A_b \eta_a) = -dC_{RT} \tag{19}$$

where

$$\begin{aligned} \eta_a &= \frac{1}{3!} \epsilon_{abcd} e^b \wedge e^c \wedge e^d, \\ C_T &= 2\left(A \wedge dA - \frac{1}{3} T^a{}_b \wedge T^b{}_c \wedge T^c{}_a\right) \\ \text{and} \quad C_{RT} &= -8(G^{ab} A_b \eta_a). \end{aligned}$$

Here $\bar{\nabla}$ is the torsion-free covariant derivative; $\mathcal{R}$, $\mathcal{R}^{ab}$ and G^{ab} are, respectively, corresponding Ricci scalar, Ricci tensor and Einstein's tensor.

Hence, we see that the $SO(4, 1)$ Pontryagin density in U_4 is the sum of four closed four forms, given by

$$F^{AB} \wedge F_{AB} = dC_R + dC_T + dC_{RT} + dC_{T\phi}. \tag{20}$$

Since all these four forms are total divergences, they yield nothing in any local theory when treated as Lagrangian densities. Hence, to have an effective field theory, however, we may consider some Lorentz invariant parts of them as Lagrangian densities. So, here we heuristically propose a Lagrangian density that combines a part of dC_{RT} with a part of $dC_{T\phi}$ as follows

$$\mathcal{L}_0 = (\mathcal{R} - \beta\phi^2)\, dT = -\frac{1}{6}(\mathcal{R} - \beta\phi^2)^* N\eta \tag{21}$$

where β is a dimensionless coupling constant.

So far $SO(3, 1)$ invariance is concerned, torsion can be separated from the connection as the torsional part of the $SO(3, 1)$ connection transforms like a tensor, i.e. when vierbeins also transform like $SO(3, 1)$ tensors in a broken $SO(4, 1)$ gauge theory. In this direction, it is important to define a torsion-free covariant differentiation through a field equation involving the connection and the vierbeins only. So we introduce Lagragian density $\mathcal{L}_1$, given by

$$\mathcal{L}_1 =^* (b_a \wedge \bar{\nabla} e^a)(b_a \wedge \bar{\nabla} e^a), \tag{22}$$

where $\bar{\nabla}$ represents covariant differentiation with respect to a $SO(3, 1)$ connection one form $\bar{\omega}^{ab}$ and b_a is a two form with one internal index and of dimension $(length)^{-1}$. If we treat b_a as Lagrange multiplier then it ensures that $\bar{\nabla}$ represents torsion-free covariant differentiation. By this way, torsion has become decoupled from the connection part of the theory. It has become independent of the one form e^a, in particular, owing to its fundamental existence as a metric-independent tensor in the affine connection in U_4, we treat here the three form $T = \frac{1}{3!} e^a \wedge T_a$ as more fundamental than the one form $T^{ab} = \omega^{ab} - \bar{\omega}^{ab}$.[2]

Now we add another Lagrangian density $\mathcal{L}_2$ containing a nonlinear kinetic term, given by

$$\mathcal{L}_2 = -f(\phi)\, d\phi \wedge^* d\phi - h(\phi)\eta \tag{23}$$

where $f(\phi)$ and $h(\phi)$ are unknown functions of ϕ whose forms are to be determined subject to the geometric structure of the manifold.

At last, we are in a position to define the total gravitational Lagrangian density in empty space, as,

$$\begin{aligned} \mathcal{L}_G &= \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2 \\ &= -\frac{1}{6}({}^*N\mathcal{R}\eta + \beta\phi^2 N) +^* (b_a \wedge \bar{\nabla} e^a)(b_a \wedge \bar{\nabla} e^a) \\ &\quad -f(\phi)\, d\phi \wedge {}^*d\phi - h(\phi)\eta, \end{aligned} \tag{24}$$

where * is Hodge duality operator, $N = 6dT$, $\mathcal{R}\eta = \frac{1}{2}\bar{R}^{ab} \wedge \eta_{ab}$. and $\eta_{ab} =^* (e_a \wedge e_b)$. To start with this Lagrangian we have altogether 69 independent components of the field variables e^a, T, $\bar{\omega}^{ab}$, ϕ and b^a. The geometrical implication of the first term, i.e. the *torsion* $\otimes$ *curvature*[3] term, in the Lagrangian $\mathcal{L}_G$ has been already discussed in Section 1.

[2] One may raise the aesthetic question of identifying T with the torsion. This can be properly addressed if we introduce two separate $SO(3, 1)$ connections ω^{ab} and $\bar{\omega}^{ab}$, and replace the Lagrangian $\mathcal{L}_2$ by the gauge invariant expression ${}^*[b_a \wedge (\overset{\omega}{\nabla} e^a - T^a)][b_a \wedge (\overset{\omega}{\nabla} e^a - T^a)] +^* [c_a \wedge (\omega^{ab} - \bar{\omega}^{ab} - T^{ab})][c_a \wedge (\omega^{ab} - \bar{\omega}^{ab} - T^{ab})]$, where the three form c_a is another Lagrange multiplier of proper dimension and $\overset{\omega}{\nabla}$ is covariant differentiation w.r.t. the connection ω.

[3] An important advantage of this part of the Lagrangian is that it is a quadratic one with respect to the field derivatives and this could be valuable in relation to the quantization program of gravity like other gauge theories of QFT.

3. EULER–LAGRANGE EQUATIONS AND GRAVITATIONAL CONSTANT

The Lagrangian $\mathcal{L}_G$ which is defined in the previous section, is only Lorentz invariant under rotation in the tangent space where de Sitter boosts are not permitted. As a consequence, T can be treated independently of e^a and $\bar{\omega}^{ab}$. Then following Hehl *et al*. (1995), we independently vary e^a, $\bar{\nabla}e^a$, dT, $\bar{R}^{ab}$, ϕ, $d\phi$ and b^a, and find

$$\delta\mathcal{L}_G = \delta e^a \wedge \frac{\partial\mathcal{L}_G}{\partial e^a} + \delta\bar{\nabla}e^a \wedge \frac{\partial\mathcal{L}_G}{\partial\bar{\nabla}e^a} + \delta dT\frac{\partial\mathcal{L}_G}{\partial dT} + \delta\bar{R}^{ab} \wedge \frac{\partial\mathcal{L}_G}{\partial\bar{R}^{ab}}$$
$$+ \delta\phi\frac{\partial\mathcal{L}_G}{\partial\phi} + \delta d\phi \wedge \frac{\partial\mathcal{L}_G}{\partial d\phi} + \delta b^a \wedge \frac{\partial\mathcal{L}_G}{\partial b^a} \tag{25}$$

$$= \delta e^a \wedge \left(\frac{\partial\mathcal{L}_G}{\partial e^a} + \bar{\nabla}\frac{\partial\mathcal{L}_G}{\partial\bar{\nabla}e^a}\right) + \delta T \wedge d\frac{\partial\mathcal{L}_G}{\partial dT} + \delta\bar{\omega}^{ab}$$
$$\wedge\left(\bar{\nabla}\frac{\partial\mathcal{L}_G}{\partial\bar{R}^{ab}} + \frac{\partial\mathcal{L}_G}{\partial\bar{\nabla}e^a} \wedge e_b\right) + \delta\phi\left(\frac{\partial\mathcal{L}_G}{\partial\phi} - d\frac{\mathcal{L}_G}{\partial d\phi}\right)$$
$$+ \delta b^a \wedge \frac{\partial\mathcal{L}_G}{\partial b^a} + d\left(\delta e^a \wedge \frac{\partial\mathcal{L}_G}{\partial\bar{\nabla}e^a} + \delta T\frac{\partial\mathcal{L}_G}{\partial dT}\right.$$
$$\left. + \delta\bar{\omega}^{ab} \wedge \frac{\partial\mathcal{L}_G}{\partial\bar{R}^{ab}} + \delta\phi\frac{\partial\mathcal{L}_G}{\partial d\phi}\right) \tag{26}$$

Using the form of the Lagrangian $\mathcal{L}_G$ given in (24), we get

$$\frac{\partial\mathcal{L}_G}{\partial e^a} = -\frac{1*}{6}N(2\mathbf{R}_a - \mathcal{R}\eta_a) -^* (b_b \wedge \bar{\nabla}e^b)^2\eta_a$$
$$- f(\phi)[-2\partial_a\phi\partial^b\phi\eta_b + \partial_b\phi\partial^b\phi\eta_a] - h(\phi)\eta_a \tag{27}$$

$$\frac{\partial\mathcal{L}_G}{\partial(\bar{\nabla}e^a)} = 2^*(b_a \wedge \bar{\nabla}e^a)b_a \tag{28}$$

$$\frac{\partial\mathcal{L}_G}{\partial(dT)} = \mathcal{R} - \beta\phi^2 \tag{29}$$

$$\frac{\partial\mathcal{L}_G}{\partial\bar{R}^{ab}} = -\frac{1}{24}{}^*N\epsilon_{abcd}e^c \wedge e^d = -\frac{1}{12}{}^*N\eta_{ab} \tag{30}$$

$$\frac{\partial\mathcal{L}_G}{\partial\phi} = -\frac{1}{3}\beta\phi N - f'(\phi)d\phi \wedge^* d\phi - h'(\phi)\eta \tag{31}$$

$$\frac{\partial\mathcal{L}_G}{\partial d\phi} = -2f^*d\phi \tag{32}$$

$$\frac{\partial \mathcal{L}_G}{\partial b^a} = 2^*(b_b \wedge \bar{\nabla} e^b)\bar{\nabla} e_a \tag{33}$$

where

$$\mathbf{R}_a := \frac{1}{2}\frac{\partial(\mathcal{R}\eta)}{de^a} = \frac{1}{4}\epsilon_{abcd}\bar{R}^{bc} \wedge e^d \tag{34}$$

and $'$ represents derivative w.r.t. ϕ.

From earlier equations, Euler–Lagrange equations for b_a gives us

$$\bar{\nabla} e^a = 0 \tag{35}$$

i.e. $\bar{\nabla}$ is torsion free. Using this result in (27) and (28) we get

$$\frac{\partial \mathcal{L}_G}{\partial e^a} = -\frac{1*}{6}N(2\mathbf{R}_a - \mathcal{R}\eta_a) - f(\phi)[-2\partial_a\phi\partial^b\phi\eta_b + \partial_b\phi\partial^b\phi\eta_a] - h(\phi)\eta_a \tag{36}$$

$$\frac{\partial \mathcal{L}_G}{\partial(\bar{\nabla} e^a)} = 0 \tag{37}$$

Hence, Euler–Lagrange equations of e^a, T and $\bar{\omega}^{ab}$, using (26), (29) and (30) give us

$$\frac{1}{6}^* N(2\mathbf{R}_a - \mathcal{R}\eta_a) + f(\phi)[-2\partial_a\phi\partial^b\phi\eta_b + \partial_b\phi\partial^b\phi\eta_a] + h(\phi)\partial\eta_a = 0 \tag{38}$$

$$d(\mathcal{R} - \beta\phi^2) = 0 \tag{39}$$

$$\bar{\nabla}(^*N\eta_{ab}) = 0 \tag{40}$$

From (31) and (32), the Euler–Lagrange equations for the field ϕ is given by

$$-\frac{1}{3}\beta\phi N + f'(\phi)\, d\phi \wedge^* d\phi - h'(\phi)\eta + 2 f d^* d\phi = 0. \tag{41}$$

Using (35) in (40)

$$d^* N = 0 \tag{42}$$

From Eqs. (39) and (42) we can write

$$^*N = \frac{6}{\kappa} \quad \text{and} \quad \mathcal{R} - \beta\phi^2 = \lambda \tag{43}$$

where κ and λ are integration constants having dimensions of $(length)^2$ and $(length)^{-2}$ respectively. Then using properties $e^a \wedge \eta_b = \delta^a{}_b\eta$ and $\mathbf{R}_a = -G^b{}_a\eta_b$ where $G^b{}_a := \mathcal{R}^b{}_a - \frac{1}{2}\mathcal{R}\delta^b{}_a$ in (38), we get

$$\mathbf{R}_a = \kappa\left[f\partial_a\phi\partial^b\phi + \frac{h}{2}\delta^b{}_a\right]\eta_b, \tag{44}$$

such that,

$$G^{b}_{\ a} = -\kappa\left[f\partial_a\phi\partial^b\phi + \frac{h}{2}\delta^{b}_{\ a}\right], \tag{45}$$

and

$$\mathcal{R}\eta = \kappa[f d\phi \wedge^* d\phi + 2h\eta]. \tag{46}$$

From (43) and (46) we get

$$\begin{aligned}\left[\frac{1}{\kappa}(\beta\phi^2 + \lambda) - 2h\right]\eta &= f d\phi \wedge^* d\phi \\ &= (f\partial_c\phi\partial^c\phi)\eta \end{aligned} \tag{47}$$

Eliminating $d\phi \wedge^* d\phi$ from (41) and (47), we get

$$\frac{2}{\kappa}\beta\phi\eta + \frac{f'}{f}\left[\frac{1}{\kappa}(\beta\phi^2 + \lambda) - 2h\right]\eta - h'(\phi)\eta + 2f d^*d\phi = 0. \tag{48}$$

4. ϕ IS LOCALIZED AT LABORATORY SCALE

Here, we study the case where ϕ is a local scalar field that vanishes at space infinity and has a quadratic Lagrangian. So we assume $f = \frac{1}{2}$, $\beta = \frac{c_\phi^2}{2}$ and $h = \textit{constant}$ in (24), where c_ϕ^2 is the dimensionless *torsion* $\times$ ϕ coupling constant, and then (45) and (48) reduce to,

$$G^{b}_{\ a} = \kappa\left[\frac{1}{2}\partial_a\phi\partial^b\phi + \frac{h}{2}\delta^{b}_{\ a}\right], \tag{49}$$

$$d^*d\phi = -\frac{1}{\kappa}c_\phi^2\phi\eta. \tag{50}$$

Using the boundary condition of ϕ at space infinity on (47) we get

$$d\phi \wedge^* d\phi = \frac{1}{\kappa}c_\phi^2\phi^2\eta. \tag{51}$$

where $\lambda = 2h\kappa$. Equation (50) is the correct field equation of a massive scalar field ϕ of mass m_ϕ, provided, we define the mass by the following equation

$$m_\phi = \frac{c_\phi}{\sqrt{\kappa}}. \tag{52}$$

This last equation shows that through the NY-term, torsion is not only connected to the gravitational constant, it also gives mass of a scalar field through the torsion $\times$ ϕ interaction term. Hence, the gravitational constant and the mass of a scalar field have the same geometrical origin in the Riemann–Cartan space U_4.

5. ϕ AND FRW COSMOLOGY

Here, we study the case where ϕ represents the local energy scale in the background of FRW cosmology. In this background, we assume ϕ to be a variable function of time only. Then, w.r.t. external indices, (45) becomes

$$G_{00} = -\kappa\left(f\dot{\phi}^2 + \frac{h}{2}g_{00}\right)$$

$$G_{ij} = -\kappa\left(\frac{h}{2}g_{ij}\right) \quad \text{where} \quad i, j = 1, 2, 3. \tag{53}$$

Here, we shall try to solve Eqs. (47) and (48) under the isotropic and homogeneous cosmological background of a universe where the metric tensor is given by the FRW metric

$$g_{00} = -1, \qquad g_{ij} = \delta_{ij}a^2(t) \quad \text{where} \quad i, j = 1, 2, 3; \tag{54}$$

such that

$$e = \sqrt{-\det(g_{\mu\nu})} = a^3 \tag{55}$$

With this assumption, Eq. (47) reduces to

$$f\dot{\phi}^2 = -\frac{1}{\kappa}(\beta\phi^2 + \lambda) + 2h. \tag{56}$$

Now, with the cosmological restriction on the metric as stated in (54) and the ϕ-field is a function of time only, Eq. (41) reduces to

$$2f\ddot{\phi} + 2f\frac{e'}{e}\dot{\phi}^2 + f'\phi^2 - \frac{2\beta}{\kappa}\phi + h' = 0 \tag{57}$$

If we eliminate $\ddot{\phi}$ from this equation with the help of the time derivative of Eq. (56), we get

$$2f\frac{e'}{e}\dot{\phi}^2 = \frac{4\beta}{\kappa}\phi - 3h' \quad \text{or,} \quad 2\frac{e'}{e} = -\frac{\frac{4\beta}{\kappa}\phi - 3h'}{\frac{1}{\kappa}(\beta\phi^2 + \lambda) - 2h} \tag{58}$$

Now, for the FRW metric, the non-vanishing components of Einstein's tensor (53) are given by

$$G^0_{\ 0} = -3\left(\frac{\dot{a}}{a}\right)^2 = -\kappa\left(\frac{\beta}{\kappa}\phi^2 + \frac{\lambda}{\kappa} - \frac{3h}{2}\right)$$

$$G^j_{\ i} = -\left(\frac{2\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right)\delta^j_{\ i} = -\kappa\frac{h}{2}\delta^j_{\ i} \tag{59}$$

Positive energy condition implies both β and λ are positive constants, and from the forms of G^0_0 and G^j_i it appears that the term $\frac{\beta}{\kappa}\phi^2$ represents pressure-less energy density, i.e. $\phi^2 \propto a^{-3} \propto \frac{1}{e}$. Putting this in (58), we get after integration

$$h = -\gamma\phi^{\frac{8}{3}} + \frac{\lambda}{2\kappa} \tag{60}$$

where γ is a constant of dimension $(length)^{\frac{-4}{3}}$. Using this functional form of h in (56) and (59), we get

$$f = -\frac{2}{3\kappa}\frac{F'}{\phi F} \quad \text{where} \quad F(\phi) = \beta\phi^2 + \frac{3}{2}\gamma\kappa\phi^{\frac{8}{3}} + \frac{\lambda}{4} \tag{61}$$

$$G^0_{\ 0} = -3\left(\frac{\dot a}{a}\right)^2 = -\kappa\left(\frac{\beta}{\kappa}\phi^2 + \frac{3\gamma}{2}\phi^{\frac{8}{3}} + \frac{\lambda}{4\kappa}\right)$$

$$G^j_{\ i} = -\left(\frac{2\dot a}{a} + \frac{\dot a^2}{a^2}\right)\delta^j_{\ i} = \kappa\left(\frac{\gamma}{2}\phi^{\frac{8}{3}} - \frac{\lambda}{4\kappa}\right)\delta^j_i \tag{62}$$

This form of G^0_0 and G^j_i implies that in the present framework, at cosmic scale, only three types of energy densities are possible, viz.

1. The pressure-less mass density $\rho_{\rm M} = \frac{\beta}{\kappa}\phi^2 \propto a^{-3}$,
2. The radiation density $\rho_{\rm R} = \frac{3\gamma}{2}\phi^{\frac{8}{3}} \propto a^{-4}$ where pressure $p_{\rm R} = \frac{1}{3}\rho_{\rm R}$ and
3. The constant vacuum energy density $\rho_{\rm V\,AC.} = \frac{\lambda}{4\kappa}$ where pressure $p_{\rm V\,AC.} = -\rho_{\rm V\,AC.}$, where β, γ and λ are all positive constants.

Hence we can write

$$G_{00} = 3H^2 = \kappa\rho,$$
$$G_{ij} = \kappa p a^2\delta_{ij} \quad \text{where} \quad i, j = 1, 2, 3; \tag{63}$$

where the Hubble's parameter $H = \dot a/a$, $\rho = \rho_{\rm M} + \rho_{\rm R} + \rho_{\rm V\,AC.}$ and $p = P_R + p_{\rm V\,AC.}$, such that ρ obeys, as a consequence of Bianchi identity $G^{\mu}_{\ 0;\mu} = 0$, the energy conservation law of Newtonian mechanics, given by the equation of state (Carrol, 2001; Padmanabhan, 2003)

$$d(\rho a^3) = -p d(a^3). \tag{64}$$

Now from (62) and (63), we get after eliminating $(\frac{\dot a}{a})^2$ that

$$\frac{\ddot a}{a} = -\frac{\kappa}{6}(\rho + 3p). \tag{65}$$

Equations (63) and (65) are two well-known results of FRW cosmology (Wald, 1984). Hence, in this background, where de Sitter gauge symmetry is broken in a Lorentz invariant way linking gravitational constant with the NY density, we have found FRW cosmology with only three kinds of energy density. Two of these kinds are that of a perfect fluid where $p = 0, \frac{\rho}{3}$ and the remaining type is that of vacuum energy where $p = -\rho$. At first glance, this result looks nothing new. In standard model $\rho = \rho_{\mathrm{M}} + \rho_{\mathrm{R}} + \rho_{\mathrm{V\ AC.}}$ is assumed empirically, but other forms of energy densities are not ruled out subject to the pressure–energy relation (64). But in our present formalism, other forms of energy densities imply different forms of the functions f and h as solutions, and this indirectly implies departure from the FRW metric at the cosmic scale. This is not the case we are studying here.

Hence, the differential equation of the evolution of the universe can be written from (65) as

$$\frac{\ddot{a}}{a} = -\frac{\kappa}{6}(\rho_{\mathrm{M}} + 2\rho_{\mathrm{R}} - 2\rho_{\mathrm{V\ AC.}}) \tag{66}$$

Using present cosmological data (Spergel *et al.*, 2003; Filippenko, 2004; Peacock, 2003), this equation ultimately implies accelerating universe. Then a reasonable dynamical age of the universe can be estimated to be 14.2 ± 1.7 Gyr. (Riess, *et al.*, 1998), consistent with the ages determined by using various other techniques (Filippenko, in press).

6. DISCUSSION

Recent cosmological evidence (Filippenko, 2004; Tonry *et al.*, 2003) suggests that cosmological constant seems to be present evermore in the cosmological data. Theoretically, cosmological constant appears when one considers a four-dimensional manifold that is due to compactification[4] of a five-dimensional manifold having the signature of a (anti)de Sitter spacetime (Padmanabhan, 2003). This implies that in the local tangent space the gauge group structure is either $SO(4, 1)$ or $SO(3, 2)$. To keep Lorentz invariance intact, (anti)de Sitter boost is forbidden in the tangent space. So it is justified, in the present contest, to consider the Lagrangian as a combination of some $SO(3, 1)$ invariant parts of the full $SO(4, 1)$ Pontryagin density.

At first, we summaries the main results obtained in this article. These are as follows.

1. The gravitational constant is related to the NY density by the relation $N = -\frac{6}{\kappa}\eta$.
2. Mass of a localised scalar field ϕ is given by the relation $m_\phi = \frac{c_\phi}{\sqrt{\kappa}}$, where c_ϕ^2 is the dimensionless *torsion* $\times$ ϕ coupling constant. By this way, we get a beautiful analogy of Coulomb's law of electro dynamics in Newtonian gravity. It can be easily checked that, with our previously described interpretation of mass, the Newtonian force between two gravitating point

[4] That is, using four-dimensional stereographic coordinates.

masses can be written as $\vec{F} = -c_1 c_2 \frac{\vec{r}}{r^3}$, where c_1^2, c_2^2 are the two respective torsional coupling constants of the corresponding masses when their dynamics is described by scalar fields in U_4.

3. When ϕ represents the local energy parameter at cosmic scale, then $\rho_{\rm M} = \frac{\beta}{\kappa}\phi^2$, $\rho_{\rm R} = \frac{3\gamma}{2}\phi^{\frac{8}{3}}$ and $\rho_{\rm V\,AC.} = \frac{\lambda}{4\kappa}$. Other kinds of energy densities are disallowed in this scenario. Here, again, β is the dimensionless *torsion* $\times$ ϕ coupling constant. Also, κ together with λ are constants of integration. γ is a constant having dimension $(length)^{-\frac{4}{3}}$. If M and V be, respectively, the total mass and volume of the universe then $\frac{\beta}{\kappa} = M^2$ and $\rho_{\rm M} = \frac{M}{V}$; this ultimately gives the local cosmological inverse length parameter $\phi = \frac{1}{\sqrt{MV}}$.

It is important to note that, in our present formalism, the only assumption is that the torsion is represented by an axial vector and the corresponding Lagrangian is a combination of two particular terms of the $SO(4, 1)$ Pontryagin density in such a way that the $SO(3, 1)$ invariance of the theory is maintained. The presence of the axial vector at each spacetime point suggests that the spacetime manifold is characterized by the presence of a 'direction vector' (vortex line) attached to each point which is the source of torsion. It may be remarked that the degrees of freedom of this theory is minimally extended from that of Einstein–Hilbert theory with torsion contributing to the additional degree. As a result, κ has got its definite geometrical meaning in U_4 space in comparison to their standard meaning of being simply constants such that κ is inversely proportional to the Nieh-Yan density. One of the remarkable features of the Lagrangian $\mathcal{L}_G$ is that $\frac{1}{\kappa}$ is not a dimensional coupling constant, $\frac{1}{\kappa}$ together with λ are constants of integration and they might have got there fixed values in the Early Universe just after the bulk matter was created when the source of gravity became able to be connected with the scalar field ϕ in the cosmological scale of a FRW universe. Further, the constancy of κ depends on the form of the source terms in $\mathcal{L}_G$ such that these terms are independent of the $SO(3, 1)$ connection. Hence, separation of the tensorial torsion part from the $SO(3, 1)$ connection, which is possible only when the $SO(4, 1)$ invariance is broken, and keeping the source independent of the $SO(3, 1)$ connection gives us constancy of κ. In other words, Lorentz invariance, in a broken de Sitter gauge theory, is associated with the constancy of κ. This constancy of κ also makes it possible to define *mass* $= \frac{c_\phi}{\sqrt{\kappa}}$ where c_ϕ^2 is the *torsion*$\times$ *matter field* coupling constant. Moreover, when we consider the metric to have the form of the FRW cosmology, then only three kinds of energy densities are possible representing mass, radiation and vacuum energy. This implies that, in this frame work, other forms of energy densities can be obtained as solutions when the metric differs from its standard FRW form. It is to be mentioned here that the scalar field ϕ of this paper is different from the Brans-Dicke scalar field.

According to Brans-Dicke theory, the value of $G = \frac{c^2\kappa}{8\pi}$ is determined by the value of the Brans-Dicke scalar field ϕ. The Brans-Dicke version of Einstein–Cartan theory, with nonzero torsion and vanishing non-metricity, was discussed by many authors (Rauch, 1984; German, 1985; Kim, 1986). In these approaches, ϕ acts as a source of torsion (Berthias, 1986). But in our approach, ϕ is connected to a local energy parameter. In laboratory scale, ϕ represents a massive scalar field where the mass arises due to *torsion* $\times$ *matter* interaction. In cosmic scale, the FRW geometry gives us $\phi = \frac{1}{\sqrt{MV}}$.

In a recent paper (Mahato, 2002), it has been shown that, in the gravity without metric formalism of gravity, when one performs a particular canonical transformation of the field variables, CP-violating θ-term appears in the Lagrangian together with the cosmological term. This supports the finding of this paper when we consider that the torsion, being an axial vector, has a certain role to play in CP violation. Indeed, the topological θ-term of 'gravity without metric formalism' is linked with the topological Nieh-Yan density of U_4 geometry. In this context, we can consider the finding of some other work (Mullick, 1995) when the gauge group is $SL(2, C)$, which is the covering group of $SO(3, 1)$, where torsion has been shown to be linked with CP violation. Thus, arrow of time plays a significant role in the geometrical origin of torsion and hence of the gravitational constant. It is to be noted here that the β-term, which is the torsion–ϕ-field interaction in the Lagrangian $\mathcal{L}_G$, ultimately gives us the mass-energy density in (61) and as $t \to \infty$ we get $G_{00}/\kappa \to$ *the constant energy density of the de Sitter space* $= \frac{\lambda}{4\kappa}$. Hence, our universe, which is presently accelerating, is heading towards a universe of constant energy density and infinite radius.

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